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LEIBNIZ HOMOLOGY AND THE HILTON–MILNOR THEOREM

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In this paper we continue the investigation into the relation between Leibniz homology and loop spaces began in [1]. Defined by Jean-Louis Loday [2, 3], Leibniz homology provides a non-commutative setting for Lie algebra homology much like cyclic homology is a non-commutative version of de Rham cohomology. In a previous paper [1] the author showed that for a group ring $\mathbf{Q}[G]$, the chain complex for the Leibniz homology of $\mathfrak{gl}(\mathbf{Q}[G])$ splits into a direct sum of complexes, each of which corresponds to a layer in the James model on BG. Here $\mathfrak{gl}(R)$ is the direct limit

$$\varinjlim_{n} \mathfrak{gl}_n(R)$$

and BG is the classifying space on G. Letting Ω denote the based loop space and Σ the reduced suspension, the James model on BG is homotopy equivalent to $\Omega\Sigma BG$ [4].

We now find a geometric interpretation of Loday's algebraic Künneth theorem for Leibniz homology. Recall that for Leibniz algebras $\mathfrak{g}$ and $\mathfrak{g}'$, Loday [5] has proved that with field coefficients

$$HL_*(\mathfrak{g} + \mathfrak{g}') \simeq HL_*(\mathfrak{g}) * HL_*(\mathfrak{g}')$$

where HL_* denotes Leibniz homology, $*$ is the free reduced product, and $+$ denotes the direct sum. On the level of spaces an identical isomorphism holds for singular homology groups,

$$H_*(\Omega\Sigma(X \vee Y)) \simeq H_*(\Omega\Sigma X) * H_*(\Omega\Sigma Y)$$

with field coefficients. Here $X \vee Y$ is the wedge of two connected, based CW-complexes. The Hilton–Milnor theorem establishes a homotopy equivalence, via iterated Samelson products, between $\Omega\Sigma(X \vee Y)$ and the Cartesian product of other loop spaces [6]. We show how a type of "Samelson product" can be defined on the chain level for Leibniz homology, which upon iteration becomes α, the inverse of Loday's Künneth theorem isomorphism. Moreover, we trace α through the geometric construction of the author's previous paper [1] and show that on Leibniz homology, α covers the map on singular homology induced by the Hilton–Milnor homotopy equivalence. In this sense, the Künneth theorem for Leibniz homology is an algebraic version of the Hilton–Milnor theorem in homotopy theory.

For further background on Leibniz homology and Leibniz algebras, the reader is referred to papers by Loday [2, 5, 7], Loday and Pirashvili [3], and Pirashvili [8]. Also see Oudom's paper [9] for additional information about the Künneth theorem. In Section 1 we discuss the algebra behind Loday's Künneth theorem, and Section 2 contains a geometric interpretation. Throughout this paper the direct sum is simply denoted as $+$.

1. THE KÜNNETH THEOREM FOR LEIBNIZ HOMOLOGY

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Leibniz algebras over a field k [3]. Then Loday [5] has shown that

$$HL_*(\mathfrak{g} + \mathfrak{g}') \simeq HL_*(\mathfrak{g}) \underset{k}{*} HL_*(\mathfrak{g}')$$

where $*$ denotes the free reduced product of the graded k-modules $HL_*(\mathfrak{g})$ and $HL_*(\mathfrak{g}')$. Note that if R and S are unital associative algebras over k, then there are natural inclusions $i: k \to R$ and $j: k \to S$. Let

$$\bar{R} = R/i[k], \qquad \bar{S} = S/j[k].$$

By definition

$$R \underset{k}{*} S = k + \bar{R} + \bar{S} + \bar{R} \otimes \bar{S} + \bar{S} \otimes \bar{R} + \bar{R} \otimes \bar{S} \otimes \bar{R} + \bar{S} \otimes \bar{R} \otimes \bar{S} + \bar{R} \otimes \bar{S} \otimes \bar{R} \otimes \bar{S}$$
$$+ \bar{S} \otimes \bar{R} \otimes \bar{S} \otimes \bar{R} + \cdots$$

where the tensor product is taken over k. The free product can be extended to graded modules

$$M = \sum_{i=0}^{\infty} M_i, \qquad N = \sum_{i=1}^{\infty} N_i,$$

with $M_0 = k$, $N_0 = k$ by using the tensor product of graded modules in the above definition. Note that

$$\bar{M} = \sum_{i=1}^{\infty} M_i, \qquad \bar{N} = \sum_{i=1}^{\infty} N_i.$$

Moreover, M is an algebra by declaring $\bar{M}$ to be a square-zero ideal, and similarly for $\bar{N}$.

Recall that $HL_*(\mathfrak{g}; k)$, the Leibniz homology of $\mathfrak{g}$ with coefficients in k [2] is given by the homology of the complex $T(\mathfrak{g})$:

$$k \overset{0}{\leftarrow} \mathfrak{g} \leftarrow \mathfrak{g}^{\otimes 2} \leftarrow \cdots \leftarrow \mathfrak{g}^{\otimes (n-1)} \overset{d}{\leftarrow} \mathfrak{g}^{\otimes n} \leftarrow \cdots .$$

Here $d: \mathfrak{g}^{\otimes n} \to \mathfrak{g}^{\otimes (n-1)}$ is defined by

$$d(g_1, g_2, \ldots, g_n) = \sum_{1 \leqslant i < j \leqslant n} (-1)^j (g_1, \ldots, g_{i-1}, [g_i, g_j], g_{i+1}, \ldots, \hat{g}_j, \ldots, g_n)$$

where $[\,,]$ is the Leibniz bracket and $(g_1, g_2, \ldots, g_n) \in \mathfrak{g}^{\otimes n}$. The chain complex for $HL_*(\mathfrak{g} + \mathfrak{g}')$ is then $T(\mathfrak{g} + \mathfrak{g}')$, where

$$TV = \sum_{n=0}^{\infty} V^{\otimes n}$$

for a vector space V. The Leibniz bracket on $\mathfrak{g} + \mathfrak{g}'$ is given by

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]).$$

We would like to view Loday's Künneth theorem as a by-product of the Hilton–Milnor theorem. The latter states that for connected CW-complexes X and Y with base vertices there is a homotopy equivalence [10, 6]

$$\theta: \Omega\Sigma X \times \Omega\Sigma(Y \vee (Y \wedge \Omega\Sigma X)) \to \Omega\Sigma(X \vee Y).$$

Moreover, Whitehead describes an explicit homotopy equivalence between these spaces [6, p. 522]. On the subspaces

$$\Omega\Sigma X \times \{*\}, \qquad \Omega\Sigma(Y \vee *),$$

θ is, up to homotopy, the inclusion map. On the factor

$$\Omega\Sigma(* \vee (Y \wedge \Omega\Sigma X))$$

θ is best interpreted in terms of the James model [4]. Consider the evident inclusion

$$Y \wedge X \hookrightarrow Y \wedge J(X) \simeq Y \wedge \Omega\Sigma X,$$

and let $\alpha_Y : Y \to \Omega\Sigma Y$ be the map $\alpha_Y(y) = \sigma_y$, where

$$\sigma_y(t) = (y, t) \in \Sigma Y.$$

Define $\alpha_X : X \to \Omega\Sigma X$ similarly, and let

$$\langle \alpha_Y, \alpha_X \rangle : Y \wedge X \to \Omega\Sigma(X \vee Y)$$

be the Samelson product [6, pp. 457, 467]. Recall that this product is first defined on the Cartesian product $f : Y \times X \to \Omega\Sigma(X \vee Y)$ by

$$f(y, x) = \alpha_Y(y)\alpha_X(x)(\alpha_Y(y))^{-1}(\alpha_X(x))^{-1}.$$

Since f restricted to $Y \vee X$ is null-homotopic, there is an induced map

$$\bar{f} : Y \wedge X \to \Omega\Sigma(X \vee Y).$$

By definition, the homotopy class of $\bar{f}$ is the Samelson product $\langle \alpha_Y, \alpha_X \rangle$. Then by using the H-space structure of $\Omega\Sigma(X \vee Y)$, $\langle \alpha_Y, \alpha_X \rangle$ extends to a map

$$J(Y \wedge X) \to \Omega\Sigma(X \vee Y)$$

which is, up to homotopy, θ restricted to $J(Y \wedge X)$. Higher iterates of the Samelson product are used to define

$$\langle \cdots \langle \langle \alpha_Y, \alpha_X \rangle, \alpha_X \rangle, \ \ldots \ \rangle : Y \wedge X^{\wedge n} \to \Omega\Sigma(X \vee Y).$$

Up to homotopy, this defines θ restricted to

$$J(Y \wedge X^{\wedge (n)}).$$

See [6] for further details. Of course, from the work of James [4] there is a quasi-isomorphism of singular chain complexes:

$$S_*(\Omega\Sigma X) \sim T(\bar{S}_*(X))$$

where $\bar{S}_*$ denotes the reduced chain complex. Furthermore,

$$S_*(\Omega\Sigma(Y \vee (Y \wedge \Omega\Sigma X))) \sim T(\bar{S}_*(Y) + \bar{S}_*(Y) \otimes \bar{T}(\bar{S}_*(X))),$$

and $S_*(\Omega\Sigma(X \vee Y)) \sim T(\bar{S}_*(X) + \bar{S}_*(Y))$, where $\sim$ denotes quasi-isomorphic. Then there is an induced map

$$\theta_* : T(A) \otimes T(B + B \otimes \bar{T}(A)) \to T(A + B)$$

where $A = \bar{S}_*(X)$, $B = \bar{S}_*(Y)$.

The goal of this section is then to define an isomorphism of chain complexes

$$\alpha : T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \to T(\mathfrak{g} + \mathfrak{g}') \tag{1}$$

which is the analogue of θ for the setting of Leibniz algebras. Note that $T(\mathfrak{g})$ and $T(\mathfrak{g} + \mathfrak{g}')$ are chain complexes on their own right with differentials $d^{\mathfrak{g}}$ an $d^{\mathfrak{g}+\mathfrak{g}'}$, respectively. Moreover,

$$T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

inherits the structure of a chain complex, since the image of the inclusion map

$$T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \hookrightarrow T(\mathfrak{g} + \mathfrak{g}')$$

is a subcomplex of $T(\mathfrak{g} + \mathfrak{g}')$. Let d^S denote the differential for

$$T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})).$$

Then the differential for

$$T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

becomes

$$d(x \otimes z) = d^{\mathfrak{g}}(x) \otimes z + (-1)^n x \otimes d^S(z)$$

where $x \in \mathfrak{g}^{\otimes n}$ and $z \in T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$. With this choice of d, however, the inclusion map of the tensor product

$$T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \hookrightarrow T(\mathfrak{g} + \mathfrak{g}')$$

is *not* map of chain complexes. To define α, we first introduce notation for elements of $T(\mathfrak{g} + \mathfrak{g}')$. The element

$$(x, 0) \in \mathfrak{g} + \mathfrak{g}'$$

will be denoted simply by x and the element

$$(0, y) \in \mathfrak{g} + \mathfrak{g}'$$

by y. Let $x_1 x_2 y_1 y_2 x_3$ denote the element

$$(x_1, 0) \otimes (x_2, 0) \otimes (0, y_1) \otimes (0, y_2) \otimes (x_3, 0) \in (\mathfrak{g} + \mathfrak{g}')^{\otimes 5}.$$

Then every element of $(\mathfrak{g} + \mathfrak{g}')^{\otimes n}$ is a k-linear combination of elements of the form $z_1 z_2 \ldots z_n$, where each $z_i = x_j$ or $z_i = y_j$.

We define α to be the identity map when restricted to the subcomplex $T(\mathfrak{g}) \otimes T(\mathfrak{g}')$ of

$$T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

i.e.,

$$\alpha(x_1 x_2 \ldots x_p \otimes y_1 y_2 \ldots y_q) = x_1 x_2 \ldots x_p y_1 y_2 \ldots y_q.$$

Since $[(x, 0), (0, y)] = 0$, we see that α thus far is a chain map. Also define α to be the identity when restricted to the subcomplex $k \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$. Letting a denote the element $(a, 0) \in \mathfrak{g} + \mathfrak{g}'$, we set

$$\alpha(x_1 \ldots x_p \otimes y_1 y_2 \ldots y_q a) = x_1 \ldots x_p \cdot \langle y_1 \ldots y_q, a \rangle$$

where

$$\langle y_1 \ldots y_q, a \rangle = y_1 \ldots y_q a + (-1)^{q+1} a y_1 \ldots y_q.$$

Then define

$$\alpha(x_1 \ldots x_p \otimes y_1 \ldots y_q a_1 \ldots a_n) = x_1 \ldots x_p \cdot \langle \cdots \langle\langle y_1 \ldots y_q, a_1\rangle, a_2\rangle, \ldots a_n\rangle$$

where

$$\langle \cdots \langle\langle y_1 \ldots y_q, a_1\rangle, a_2\rangle, \ldots a_n\rangle = \langle \cdots \langle\langle y_1 \ldots y_q, a_1\rangle, a_2\rangle, \ldots a_{n-1}\rangle \cdot a_n + (-1)^{n+q} a_n \cdot \langle \cdots \langle\langle y_1 \ldots y_q, a_1\rangle, a_2\rangle, \ldots a_{n-1}\rangle \cdot$$

Of course, α is extended to be a k-linear map.

LEMMA. 1.1. *For $\omega = x_1 x_2 \ldots x_p \otimes y_1 \ldots y_p \cdot a_1 \ldots a_n$, we have*

$$\alpha \circ d(\omega) = d^{\mathfrak{g}+\mathfrak{g}'} \circ \alpha(\omega)$$

where d is the differential for $T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$.

Proof. The proof follows by induction on n. The case $n = 1$ can be checked by hand. Now,

$$\begin{aligned} d^{\mathfrak{g}+\mathfrak{g}'} \alpha(\omega) &= [d^{\mathfrak{g}+\mathfrak{g}'} \alpha(x_1 \ldots x_p \otimes y_1 \ldots y_q a_1 \ldots a_{n-1}] \cdot a_n \\ &+ (-1)^{n+q} d^{\mathfrak{g}}(x_1 \ldots x_p) \cdot \alpha(a_n \otimes y_1 \ldots y_q a_1 \ldots a_{n-1}) \\ &+ (-1)^{n+p+q} x_1 \ldots x_p \cdot d^{\mathfrak{g}+\mathfrak{g}'}(\alpha(a_n \otimes y_1 \ldots y_q a_1 \ldots a_{n-1})) \\ &+ \sum_{i=1}^{n-1} (-1)^{p+q+n} x_1 \ldots x_p \cdot \langle \cdots \langle \cdots \langle y_1 \ldots y_q, a_1\rangle \cdots [a_i, a_n]\rangle \cdots \rangle \cdot \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha d(\omega) &= [\alpha d(x_1 \ldots x_p \otimes y_1 \ldots y_q a_1 \ldots a_{n-1}] \cdot a_n \\ &+ (-1)^{n+q} d^{\mathfrak{g}}(x_1 \ldots x_p) \cdot \alpha(a_n \otimes y_1 \ldots y_q a_1 \ldots a_{n-1}) \\ &+ (-1)^{n+p+q+1} x_1 \ldots x_p \cdot \alpha(a_n \otimes d^s(y_1 \ldots y_q a_1 \ldots a_{n-1})) \\ &+ \sum_{i=1}^{n-1} (-1)^{p+q+n} x_1 \ldots x_p \cdot \langle \cdots \langle \cdots \langle y_1 \ldots y_q, a_1\rangle \cdots [a_i, a_n]\rangle \cdots \rangle \end{aligned}$$

where d^s is the differential for $T(\mathfrak{g} + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$. By the induction hypothesis, the first terms in both of these sums are equal. Moreover,

$$\begin{aligned} d^{\mathfrak{g}+\mathfrak{g}'} \alpha(a_n \otimes y_1 \ldots y_q a_1 \ldots a_{n-1}) &= \alpha d(a_n \otimes y_1 \ldots y_q a_1 \ldots a_{n-1}) \\ &= -\alpha(a_n \otimes d^s(y_1 \ldots y_q a_1 \ldots a_{n-1})). \end{aligned}$$

It follows that $\alpha d(\omega) = d^{\mathfrak{g}+\mathfrak{g}'} \alpha(\omega)$. □

Let u be a monomial in the x_i's and y_j's in $T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$. Define α inductively by

$$\alpha(x_1 \ldots x_p \otimes ub) = \begin{cases} \alpha(x_1 \ldots x_p \otimes u) \cdot b & \text{for } (0, b) \in \mathfrak{g} + \mathfrak{g}' \\ x_1 \ldots x_p \cdot \langle \kappa(u), b\rangle & \text{for } (b, 0) \in \mathfrak{g} + \mathfrak{g}'. \end{cases}$$

Here $\kappa(u)$ is defined by the equation

$$\alpha(x_1 \ldots x_p \otimes u) = x_1 \ldots x_p \cdot \kappa(u).$$

The operation $\langle\,,\rangle$ is bilinear with

$$\langle u, b\rangle = u\cdot b - (-1)^{|u|} b\cdot u$$

where $|u|$ denotes the length of u. If $u = u_1 u_2 \ldots u_n$ with each $u_i = x_j$ or $u_i = y_j$, then $|u| = n$.

LEMMA 1.2. *With α defined as above, we have*

$$\alpha d = d^{\mathfrak{g}+\mathfrak{g}'}\alpha.$$

Proof. The proof follows by induction on the length of u. If b represents the element $(0, b) \in \mathfrak{g} + \mathfrak{g}'$, then

$$d^{\mathfrak{g}+\mathfrak{g}'}\alpha(x_1 \ldots x_p \otimes ub) = I + II$$

where $I = d^{\mathfrak{g}+\mathfrak{g}'}(\alpha(x_1 \ldots x_p \otimes u))\cdot b$, and II represents the sum of bracket terms involving b and any $\mathfrak{g}'$ factor of $\alpha(x_1 \ldots x_p \otimes u)$. On the other hand,

$$\alpha d(x_1 \ldots x_p \otimes ub) = I' + II'$$

$$I' = \alpha(d(x_1 \ldots x_p \otimes u))\cdot b$$

$$II' = \alpha\left(\sum_{i=1}^{n} (-1)^{p+i} x_1 \ldots x_p \otimes u_1 \ldots [u_i, b] \ldots u_n\right).$$

By the induction hypothesis, $I = I'$. One checks that $II = II'$. If b represents the element $(b, 0) \in \mathfrak{g} + \mathfrak{g}'$, then the proof of Lemma 1.1 can be used to show that

$$\alpha d = d^{\mathfrak{g}+\mathfrak{g}'}\alpha.$$ □

In [5] Loday has essentially constructed a chain map

$$h: T(\mathfrak{g}+\mathfrak{g}') \to T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

which we shall show is both a left and right inverse of the "Samelson product" map α. It then follows that α is an isomorphism of chain complexes. Let h be the identity map on both subcomplexes $T(\mathfrak{g})$ and $T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$ of $T(\mathfrak{g}+\mathfrak{g}')$. Loday denotes by $C_{\mathfrak{g}}$ the subcomplex of $T(\mathfrak{g}+\mathfrak{g}')$ given by

$$\mathfrak{g}^{\otimes n} \otimes (\mathfrak{g}')^{\otimes m} \otimes \mathfrak{g}^{\otimes p} \ldots, n \geq 1 \text{ and } m \geq 1.$$

It follows quickly that $C_{\mathfrak{g}}$ can be identified with

$$\bar{T}(\mathfrak{g}) \otimes \bar{T}(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

as k-modules. Loday defines a map of chain complexes

$$h|_{C_{\mathfrak{g}}}: C_{\mathfrak{g}} \to \bar{T}(\mathfrak{g}) \otimes \bar{T}(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

by $h = \sum_{k\geq 0} h^{\langle k\rangle}$. If

$$z = x_1 \ldots x_n \cdot y_1 \ldots y_m \cdot x_{n+1} \ldots x_{n+p} \cdot \ldots$$

then

$$h^0(z) = x_1 \ldots x_n \otimes y_1 \ldots y_m \cdot x_{n+1} \ldots x_{n+p} \cdot \ldots .$$

For $k \geqslant 1$,

$$h^{\langle k \rangle}(z) = \sum_{n < i_1 < \ldots < i_k} \operatorname{sgn}(\sigma) x_1 \ldots x_n x_{i_1} x_{i_2} \ldots x_{i_k} \otimes y_1 \ldots y_m \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots$$

where the sum is extended over all k-tuples of x's in $\mathfrak{g} + 0$ beyond x_n. Here σ is the evident permutation which sends the factors of z to

$$x_1 x_2 \ldots x_n x_{i_1} \ldots x_{i_k} y_1 \ldots y_m \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots .$$

The function h is then a well-defined chain map

$$h : T(\mathfrak{g} + \mathfrak{g}') \to T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})).$$

THEOREM 1.3. *The composition $h \circ \alpha$ is the identity on*

$$T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

and $\alpha \circ h$ is the identity on $T(\mathfrak{g} + \mathfrak{g}')$.

Proof. Let us first show that $h \circ \alpha = \mathbf{1}$. Clearly, $h \circ \alpha = \mathbf{1}$ on the subcomplexes $T(\mathfrak{g}) \otimes T(\mathfrak{g}')$ and $k \otimes T(\mathfrak{g} + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$ of $T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$. Let

$$z = x_1 \ldots x_p \otimes u \in T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

where u is a monomial in x_i's and y_j's. The proof proceeds by induction on the length of u. If $|u| = 1$, then $u = y$ for some $y \in \mathfrak{g}'$, and $(h \circ \alpha)(z) = z$. Suppose that $(h \circ \alpha)(z) = z$ for $|u| = n$. If b represents the element $(b, 0) \in \mathfrak{g} + \mathfrak{g}'$, then

$$h \circ \alpha(x_1 \ldots x_p \otimes ub) = h(\alpha(x_1 \ldots x_p \otimes u)b) + (-1)^{|u|-1} h(x_1 \ldots x_p b \cdot \kappa(u))$$

where $\kappa(u)$ is defined by

$$\alpha(x_1 \ldots x_p \otimes u) = x_1 \ldots x_p \cdot \kappa(u).$$

Now,

$$\begin{aligned} h(\alpha(x_1 \ldots x_p \otimes u)b) &= (h\alpha(x_1 \ldots x_p \otimes u))b + (-1)^{|u|} h(x_1 \ldots x_p b \cdot \kappa(u)) \\ &= x_1 \ldots x_p \otimes ub + (-1)^{|u|} h(x_1 \ldots x_p b \cdot \kappa(u)). \end{aligned}$$

It follows that

$$h \circ \alpha(x_1 \ldots x_p \otimes ub) = x_1 \ldots x_p \otimes ub.$$

If b represents the element $(0, b) \in \mathfrak{g} + \mathfrak{g}'$, then

$$\begin{aligned} h \circ \alpha(x_1 \ldots x_p \otimes ub) &= h(\alpha(x_1 \ldots x_p \otimes u)b) \\ &= (h\alpha(x_1 \ldots x_p \otimes u))b \\ &= x_1 \ldots x_p \otimes ub. \end{aligned}$$

By work of Loday [5], h is known to be an isomorphism. Thus it follows that $\alpha \circ h = \mathbf{1}$. It can also be checked combinatorially that $\alpha \circ h = \mathbf{1}$, which would give an alternate proof that h is in fact an isomorphism of chain complexes. □

It follows that

$$\alpha : T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \to T(\mathfrak{g} + \mathfrak{g}')$$

is an isomorphism of chain complexes. The domain of α can be written as the direct sum of complexes

$$k + \bar{T}(\mathfrak{g}) + \bar{T}(W) + \bar{T}(\mathfrak{g}) \otimes \bar{T}(W)$$

where $W = \mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})$. To compare this result with Loday's [5], note that $\bar{T}(\mathfrak{g}) \otimes \bar{T}(W)$ can be identified with the subcomplex $C\mathfrak{g}$ of $T(\mathfrak{g} + \mathfrak{g}')$. Moreover, $\bar{T}(W)$ is isomorphic as a chain complex to the direct sum

$$\bar{T}(\mathfrak{g}') + C(\mathfrak{g}')$$

where $C(\mathfrak{g}')$ is the subcomplex of $T(\mathfrak{g} + \mathfrak{g}')$ given by

$$(\mathfrak{g}')^{\otimes n} \otimes \mathfrak{g}^{\otimes m} \otimes (\mathfrak{g}')^{\otimes p} \ \dots, n \geqslant 1, m \geqslant 1.$$

Arranging the tensor products as in Loday's paper [5], and applying the classical Künneth theorem, one has

$$HL_*(\mathfrak{g} + \mathfrak{g}') \simeq HL_*(\mathfrak{g}) \underset{k}{*} HL_*(\mathfrak{g}')$$

where k is a field.

2. AN APPLICATION TO THE RING OF INFINITE MATRICES

In this section we apply the Künneth theorem to the calculation of

$$HL_*(\mathfrak{gl}(R) + \mathfrak{gl}(S))$$

where R and S are unital algebras over $\mathbf{Q}$. Recall that Cuvier [11] and Loday [2] have shown that for a unital algebra A over $\mathbf{Q}$,

$$HL_*(\mathfrak{gl}(A)) \simeq T(HH_{*-1}(A))$$

where HH_{*-1} are the Hochschild homology groups shifted by one dimension. As rings,

$$\mathfrak{gl}(R) + \mathfrak{gl}(S) \simeq \mathfrak{gl}(R + S)$$

where $+$ denotes the direct sum. Thus, there are two approaches to the calculation of $HL_*(\mathfrak{gl}(R) + \mathfrak{gl}(S))$, one in terms of the factors $HL_*(\mathfrak{gl}(R))$, $HL_*(\mathfrak{gl}(S))$ suggested by the Künneth theorem, and the other as the direct computation of

$$HL_*(\mathfrak{gl}(R + S)).$$

We wish to compare these calculations via the "Samelson product" map α introduced in Section 1. For any $\mathbf{Q}$-algebra A, the first step in the calculation of $HL_*(\mathfrak{gl}(A))$ is to form the quotient of the chain complex for Leibniz homology by the adjoint action of $\mathfrak{gl}(\mathbf{Q})$. See [11, 2]. Using invariant theory, the resulting quasi-isomorphic complex is denoted $L_*(A)$:

$$\mathbf{Q}[\Sigma_1] \underset{\mathbf{Q}}{\otimes} A \xleftarrow{d} \mathbf{Q}[\Sigma_2] \underset{\mathbf{Q}}{\otimes} A^{\otimes 2} \xleftarrow{d} \cdots \xleftarrow{d} \mathbf{Q}[\Sigma_{n+1}] \underset{\mathbf{Q}}{\otimes} A^{\otimes (n+1)} \leftarrow \cdots . \tag{2}$$

Here Σ_n denotes the symmetric group on n letters. See [2, 1] for an explicit description of the boundary map d. In this section we compare the complexes $L_*(R)$, $L_*(S)$, $L_*(R + S)$ by analyzing the maps induced by α and h after applying invariant theory.

We first simplify the chain complex $L_*(R+S)$ by exploiting the isomorphism [2, p. 14]

$$HH_*(R+S) \simeq HH_*(R) + HH_*(S).$$

Let $N_*^{cy}(R)$ be the chain complex for the Hochschild homology of R [2, p. 9]. Often $N_*^{cy}(R)$ is called the cyclic bar construction on R [2], and

$$N_n^{cy}(R) = R^{\otimes(n+1)}.$$

There is an inclusion map of complexes

$$\imath : N_*^{cy}(R) + N_*^{cy}(S) \to N_*^{cy}(R+S)$$
$$\imath : R^{\otimes(n+1)} + S^{\otimes(n+1)} \to (R+S)^{\otimes(n+1)}$$

and a projection map of complexes

$$p : N_*^{cy}(R+S) \to N_*^{cy}(R) + N_*^{cy}(S)$$
$$p : (R+S)^{\otimes(n+1)} \to R^{\otimes(n+1)} + S^{\otimes(n+1)}$$

where $\ker p = \sum_{k=1}^{n} R^{\otimes k} \otimes S^{\otimes(n+1-k)}$. Then $p \circ \imath = \mathbf{1}$ on $N_*^{cy}(R) + N_*^{cy}(S)$, and $\imath \circ p$ is chain homotopy equivalent to the identity on $N_*^{cy}(R+S)$, [2, pp. 14, 21]. We denote a monomial in $(R+S)^{\otimes(n+1)}$ by $(x_0, x_1, \ldots, x_n)$, where each x_i represents the element $(x_i, 0) \in R+S$ or the element $(0, x_i) \in R+S$. Consider elements

$$\sigma \otimes (x_0, x_1, \ldots, x_n) \in \mathbf{Q}[\Sigma_{n+1}] \otimes (R+S)^{\otimes(n+1)},$$

where for each cycle $(\omega_0, \omega_1, \ldots, \omega_p)$ of σ, the elements

$$x_{\omega_0}, x_{\omega_1}, \ldots, x_{\omega_p}$$

are either all in R or all in S. When we refer to a cycle decomposition of σ, we, of course, mean a decomposition into disjoint cycles. Let $M_n(R+S)$ denote the vector space spanned by $\mathbf{Q}$-linear combinations of such $\sigma \otimes (x_0, x_1, \ldots, x_n)$'s. Under the boundary map d of $L_*(R+S)$, only those x_i's which correspond to the same cycle are multiplied together [2, 1]. It follows that $M_*(R+S)$ is a subcomplex $L_*(R+S)$, and there is an inclusion map of complexes

$$\imath : M_*(R+S) \to L_*(R+S).$$

We define a projection map

$$\pi : L_*(R+S) \to M_*(R+S)$$

of complexes as follows. Recall that for an algebra A, $L_*(A)$ splits as a direct sum of complexes $\sum_{m\geqslant 1} P_*^{(m)}$, where each $P_*^{(m)}$ is isomorphic to the (level-wise) tensor product of certain (pre)-multisimplicial k-modules [1] which have the form

$$\mathbf{Q}[\Sigma_*^{(m)}] \otimes N^{cy}(A)^m.$$

Here $\Sigma^{(m)}$ is the family of all permutations with precisely m-many cycles and $N^{cy}(A)^m$ is the m-fold tensor product of the chain complex $N^{cy}(A)$. In multisimplicial dimension $q_1, q_2, \ldots, q_m$,

$$\sigma \otimes \vec{a} \in (\mathbf{Q}[\Sigma_*^{(m)}] \otimes N^{cy}(A)^m)_{(q_1, q_2, \ldots, q_m)}$$

if

$$\sigma = (w_0 \ldots w_{q_1})(w_{q_1+1} \ldots w_{q_1+q_2+1}) \ldots (w_{N-q_m} \ldots w_N)$$
$$\vec{a} = (a_0, a_1, \ldots, a_{q_1}) \otimes (a_{q_1+1}, \ldots, a_{q_1+q_2+1}) \otimes \ldots \otimes (a_{N-q_m}, \ldots, a_N)$$

where $N = q_1 + q_2 + \cdots + q_m + m - 1$. The projection map

$$p: N_*^{cy}(R + S) \to N_*^{cy}(R) + N_*^{cy}(S)$$

can be iterated to define another projection

$$1 \otimes p^{\otimes m}: \mathbf{Q}[\Sigma^{(m)}] \otimes N^{cy}(R + S)^m \to \mathbf{Q}[\Sigma^{(m)}] \otimes [N^{cy}(R) + N^{cy}(S)]^{\otimes m}.$$

Taking the direct sum over m, the above map yields

$$\pi: L_*(R + S) \to M_*(R + S).$$

Note that $M_*(R + S)$ can be identified with

$$\sum_{m \geqslant 1} \mathbf{Q}[\Sigma_*^{(m)}] \otimes [N^{cy}(R) + N^{cy}(S)]^{\otimes m}$$

where we again use the level-wise tensor product of multisimplicial objects.

LEMMA 2.2. *The maps* $\iota: M_*(R + S) \to L_*(R + S)$ *and* $\pi: L_*(R + S) \to M_*(R + S)$ *induce isomorphisms on homology with integer coefficients.*

Proof. From [1, 2.10] we know that

$$H_n(\mathbf{Z}[\Sigma_*^{(m)}]; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & n = 0 \\ 0, & n \geqslant 1. \end{cases}$$

Since the inclusion map $N^{cy}(R) + N^{cy}(S) \to N^{cy}(R + S)$ induces an isomorphism

$$HH_*(R; \mathbf{Z}) + HH_*(S; \mathbf{Z}) \to HH_*(R + S; \mathbf{Z})$$

if follows that

$$\iota_*: H_*(M_*(R + S); \mathbf{Z}) \to H_*(L_*(R + S); \mathbf{Z})$$

is an isomorphism. A similar statement applies to the projection

$$N^{cy}(R + S) \to N^{cy}(R) + N^{cy}(S).$$

Of course, the lemma also holds with rational coefficients. □

Let $\mathfrak{g} = \mathfrak{gl}(R)$ and $\mathfrak{g}' = \mathfrak{gl}(S)$. We now describe the complex

$$T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$$

with the aid of invariant theory. The chain complex $T(\mathfrak{g})$ is quasi-isomorphic to $\mathbf{Q} + L_*(A)$, where $\mathbf{Q}$ is in dimension zero. Let C_* be the image of the composition of chain maps $\pi \circ q \circ j$,

$$T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \xrightarrow{j} T(\mathfrak{g} + \mathfrak{g}') \xrightarrow{q} \mathbf{Q} + L_*(R + S) \xrightarrow{\pi} \mathbf{Q} + M_*(R + S)$$

where j is the inclusion map, and q is constructed from the quotient map

$$\mathfrak{gl}(R + S)^{\otimes n} \to \mathfrak{gl}(R + S)^{\otimes n}/\mathfrak{gl}(\mathbf{Q}) \simeq \mathbf{Q}[\Sigma_n] \otimes (R + S)^{\otimes n}.$$

The Lie algebra $\mathfrak{gl}(\mathbf{Q})$ acts on $\mathfrak{gl}(R + S)^{\otimes n}$ via the adjoint action [2, p. 281; 12].

LEMMA 2.2. *With rational coefficients*

$$H_*(C_*; \mathbf{Q}) \simeq H_*(T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))).$$

Proof. By the results of Section 1, j is injective on homology. Furthermore, q and π induce isomorphisms on homology. Rational coefficients are needed to insure that the quotient map q is a quasi-isomorphism [2, p. 326]. □

There is an alternative description of C_* using tensor algebras of R and S. Let

$$T^{(m)}(S + S \otimes \bar{T}(R))$$

be the $\mathbf{Q}$-vector subspace of $T(S + S \otimes \bar{T}(R))$ spanned by monomials $(x_1, x_2, \dots, x_m)$, where $x_1 \in S$ and in general either $x_i \in S$ or $x_i \in R$ for each i. Then C_* is the vector subspace of

$$\sum_{m \geq 1} \mathbf{Q}[\Sigma_m] \otimes T^{(m)}(S + S \otimes \bar{T}(R))$$

spanned by elements of the form

$$\sigma \otimes (x_1, x_2, \dots, x_m)$$

where for each cycle $(\omega_0, \omega_1, \dots, \omega_p)$ of σ, the entries $x_{\omega_0}, x_{\omega_1}, \dots, x_{\omega_p}$ are either all in R or all in S.

The map $h: T(\mathfrak{g} + \mathfrak{g}') \to T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$ induces a quotient chain map

$$h_\Sigma: \mathbf{Q} + M_*(R + S) \to (\mathbf{Q} + L_*(R)) \otimes C_*.$$

For example, let

$$z = (0\,1)(2)(3\,4)(r_0, r_1, s_0, r_2, r_3) \in M_5(R + S).$$

Then

$$\begin{aligned} h_\Sigma(z) = {} & (0\,1)(r_0, r_1) \otimes (0)(12)(s_0, r_2, r_3) - (0\,1)(2)(r_0, r_1, r_2) \otimes (0)(1)(s_0, r_3) \\ & + (0\,1)(2)(r_0, r_1, r_3) \otimes (0)(1)(s_0, r_2) + (0\,1)(2\,3)(r_0, r_1, r_2, r_3) \otimes (0)(s_0). \end{aligned}$$

The Samelson product isomorphism

$$\alpha: T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \to T(\mathfrak{g} + \mathfrak{g}')$$

can be used to define an interesting chain map

$$\alpha_\Sigma: (\mathbf{Q} + L_*(R)) \otimes C_* \to \mathbf{Q} + M_*(R + S)$$

which involves the action of Σ_n on itself via conjugation. One must first choose a lifting

$$\rho: (\mathbf{Q} + L_*(R)) \otimes C_* \to T(\mathfrak{g}) \otimes T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})).$$

Let

$$\sigma \cdot u \in \mathbf{Q}[\Sigma_n] \otimes R^{\otimes n}$$

$$\tau \cdot w \in \mathbf{Q}[\Sigma_m] \otimes T^{(m)}(S + S \otimes \bar{T}(R)).$$

Let $E^c_{i,j}$ denote the elementary matrix with c in the i, j position and zeroes everywhere else. Then

$$\begin{aligned}\rho(\sigma\cdot u\otimes\tau\cdot w) &= (E^{u_1}_{1,\sigma(1)}, E^{u_2}_{2,\sigma(2)}, \dots, E^{u_n}_{n,\sigma(n)})\\ &\otimes (E^{w_1}_{n+1,\tau(1)+n}, E^{w_2}_{n+2,\tau(2)+n}, \dots, E^{w_m}_{n+m,\tau(m)+n}).\end{aligned} \tag{3}$$

Furthermore, let $\sigma\times\tau$ be the element of Σ_{n+m} given by the evident monomorphism

$$\Sigma_n\times\Sigma_m \hookrightarrow \Sigma_{n+m}.$$

Then $\rho(\sigma\cdot u\otimes\tau\cdot w)$ is the usual lifting of $(\sigma\times\tau)\cdot(u\otimes w)$ from invariant theory [2, p. 307]. Now α_Σ is defined as the composition $q'\circ\alpha\circ\rho$, where

$$q' : T(\mathfrak{g}+\mathfrak{g}')\to \mathbf{Q}+M_*(R+S)$$

is the quotient map $\pi\circ q$ defined previous to Lemma 2.2. Although $d\rho\neq\rho d$ for the corresponding boundary maps d, we do have that

$$dq'\alpha\rho = q'd\alpha\rho = q'\alpha d\rho = q'\alpha\rho d.$$

The last equality, $q'\alpha d\rho = q'\alpha\rho d$ holds, since $\alpha d\rho$ and $\alpha\rho d$ represent the same element in $T(\mathfrak{g}+\mathfrak{g}')$ modulo the adjoint action of $\mathfrak{gl}(\mathbf{Q})$. It follows that α_Σ is a chain map.

LEMMA 2.3. *With ρ as in (3), the induced map*

$$\alpha_\Sigma : (\mathbf{Q}+L_*(R))\otimes C_*\to\mathbf{Q}+M_*(R+S)$$

is given by

$$\alpha_\Sigma(\sigma\cdot u\otimes\tau\cdot w) = \sum_\gamma \pm[\gamma^{-1}(\sigma\times\tau)\gamma]\cdot(u\otimes w),$$

where the sum is over all permutations γ of

$$(u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m)$$

which occur in the original definition of α. Here γ acts on the entries of $u\otimes w$ by place permutation.

Proof. The proof follows from [2, p. 282, Lemma (9.2.8)], which states that Weyl's isomorphism of invariant theory [13]

$$\mathfrak{gl}(\mathbf{Q})^{\otimes n}/\mathfrak{gl}(\mathbf{Q}) \xrightarrow{\simeq} \mathbf{Q}[\Sigma_n]$$

is Σ_n-equivariant. The action of Σ_n on $\mathfrak{gl}(\mathbf{Q})^{\otimes n}$ is by place permutation and Σ_n acts on itself by conjugation. □

LEMMA 2.4. *The map α_Σ induces an isomorphism on homology,*

$$(\alpha_\Sigma)_* : H_*((\mathbf{Q}+L_*(R))\otimes C_*)\to H_*(\mathbf{Q}+M_*(R+S))$$

with rotational coefficients.

Proof. Recall that $\alpha_\Sigma = q'\circ\alpha\circ\rho$, where α and q' are known to induce homology isomorphisms. Furthermore, over the rationals,

$$H_*(T(\mathfrak{g})\otimes T(\mathfrak{g}'+\mathfrak{g}'\otimes\bar{T}(\mathfrak{g}))) \simeq H_*(T(\mathfrak{g}))\otimes H_*(T(\mathfrak{g}'+\mathfrak{g}'\otimes\bar{T}(\mathfrak{g})))$$

and

$$H_*(T(\mathfrak{g})) \simeq H_*(\mathbf{Q} + L_*(R))$$

$$H_*(T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))) \simeq H_*(C_*).$$

Let a be a cycle in $T(\mathfrak{g})$ which represents a homology class in dimension $(n-1)$, and let b be a cycle in $T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))$ representing a homology class in dimension $(m-1)$. Since the quotient by the adjoint action of $\mathfrak{gl}(\mathbf{Q})$ on $T(\mathfrak{g})$ induces an isomorphism on homology [2, p. 326], a may be chosen to be a finite sum $\sum_i \lambda_i a_i$, where $\lambda_i \in \mathbf{Q}$ and each a_i is an n-fold tensor product of elementary matrices:

$$E^{u_1}_{1,\sigma(1)} \otimes E^{u_2}_{2,\sigma(2)} \otimes \cdots \otimes E^{u_n}_{n,\sigma(n)}$$

and $\sigma \in \Sigma_n$, $u_i \in R$. Similarly b may be chosen to be a finite sum $\sum_j \mu_j b_j$, where $\mu_j \in \mathbf{Q}$ and each b_j is an m-fold tensor product of elementary matrices:

$$E^{w_1}_{n+1,\tau(1)+n} \otimes E^{w_2}_{n+2,\tau(2)+n} \otimes \cdots \otimes E^{w_m}_{n+m,\tau(m)+n}$$

where $\tau \in \Sigma_m$, $w_i \in (R \cup S)$. Then

$$\rho\left(\left(\sum_i \lambda_i \sigma \otimes (u_1, u_2, \ldots, u_n)_{(i)}\right) \otimes \left(\sum_j \mu_j \sigma \otimes (w_1, w_2, \ldots, w_m)_{(j)}\right)\right) = \left(\sum_i \lambda_i a_i\right) \otimes \left(\sum_j \mu_j b_j\right)$$
$$= a \otimes b \qquad \square$$

To give these calculations a more geometric flavor, recall [1] that for a group ring $\mathbf{Q}[G]$, we have

$$HL_*(\mathfrak{gl}(\mathbf{Q}[G])) \simeq H_*(\Omega\Sigma\Sigma_+(\Lambda BG); \mathbf{Q}),$$

where $\Lambda BG = \mathrm{Maps}(S^1, BG)$ is the free loop space on BG and Σ_+ denotes the unreduced suspension. For groups G and H there are isomorphisms

$$\begin{aligned} HL_*(\mathfrak{gl}(\mathbf{Q}[G]) + \mathfrak{gl}(\mathbf{Q}[H])) &\simeq HL_*(\mathfrak{gl}(\mathbf{Q}[G])) * HL_*(\mathfrak{gl}(\mathbf{Q}[H])) \\ &\simeq H_*(\Omega\Sigma\Sigma_+(\Lambda BG); \mathbf{Q}) * H_*(\Omega\Sigma\Sigma_+(\Lambda BH); \mathbf{Q}) \\ &\simeq H_*(\Omega\Sigma(\Sigma_+ BG \vee \Sigma_+ BH); \mathbf{Q}). \end{aligned}$$

LEMMA 2.5. *Let R and S be unital $\mathbf{Q}$-algebras. Then with rational coefficients*

$$H_*(C_*; \mathbf{Q}) \simeq T(HH_{*-1}(S) + HH_{*-1}(S) \otimes \bar{T}(HH_{*-1}(R)))$$

where C_* is the image of the composition $\pi \circ q \circ j$,

$$T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g})) \xrightarrow{j} T(\mathfrak{g} + \mathfrak{g}') \xrightarrow{q} \mathbf{Q} + L_*(R+S) \xrightarrow{\pi} \mathbf{Q} + M_*(R+S)$$

$\mathfrak{g} = \mathfrak{gl}(R)$, $\mathfrak{g}' = \mathfrak{gl}(S)$.

Proof. The proof uses the alternative description of C_* given after Lemma 2.2 along with techniques given in [1, 2.11, 2.12] First write C_* as a direct sum of complexes $\sum_{m \geq 1} P^{(m)}$, where each $P^{(m)}$ is a chain complex associated to the m-cycles in the family of symmetric groups. The homology groups $H_*(P^{(m)})$ can be computed by showing that $P^{(m)}$ is quasi-isomorphic to the tensor product certain cyclic bar constructions. $\square$

COROLLARY 2.6. *Let* $R = \mathbf{Q}[G]$, $S = \mathbf{Q}[H]$, $X = \Sigma_+(\Lambda BG)$, $Y = \Sigma_+(\Lambda BH)$. *Then with rational coefficients*

$$H_*(C_*; \mathbf{Q}) \simeq H_*(J(Y \vee (Y \wedge JX)))$$

where J denotes the James model.

Proof. Recall [14] that with integer coefficients

$$H_*(\Lambda BG; \mathbf{Z}) \simeq HH_*(\mathbf{Z}[G]; \mathbf{Z})$$

$$\bar{H}_*(\Sigma_+(\Lambda BG); \mathbf{Z}) \simeq HH_{*-1}(\mathbf{Z}[G]; \mathbf{Z}).$$

The result now follows from Lemma 2.5. □

Using singular homology of spaces, α_Σ induces a rational isomorphism

$$H_*(JX) \otimes H_*(J(Y \vee (Y \wedge JX))) \to H_*(J(X \vee Y))$$

where X and Y are defined in Corollary 2.6. Since there is an isomorphism

$$H_*(\mathbf{Q} + L_*(R); \mathbf{Q}) \simeq H_*(JX; \mathbf{Q})$$

(and similarly with JY), we may choose an isomorphism ζ so that the following diagram commutes with rational coefficients:

$$\begin{tikzcd}
HL_*(\mathfrak{g}) \otimes H_*(T(\mathfrak{g}' + \mathfrak{g}' \otimes \bar{T}(\mathfrak{g}))) \arrow[r, "\alpha_*"] \arrow[d, "q_* \otimes q'|_*"', "\simeq"] & HL_*(\mathfrak{g} + \mathfrak{g}') \arrow[d, "q'_*"', "\simeq"] \\
H_*(\mathbf{Q} + L_*(R)) \otimes H_*(C_*) \arrow[r, "(\alpha_\Sigma)_*"] \arrow[d, "\zeta"', "\simeq"] & H_*(\mathbf{Q} + M_*(R + S)) \arrow[d, "\simeq"] \\
H_*(JX) \otimes H_*(J(Y \vee (Y \wedge JX))) \arrow[r, "\theta_*"] & H_*(J(X \vee Y)).
\end{tikzcd}$$

Here θ_* is the isomorphism induced by the homotopy equivalence θ of the Hilton–Milnor theorem, and α_* is the inverse of Loday's Künneth theorem isomorphism. Although ζ is not given explicitly, the diagram summarizes the relation between the Künneth and Hilton–Milnor theorems.

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